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LETTERS TO THE EDITOR

SUBHARMONIC MELNIKOV FUNCTIONS FOR STRONGLY ODD NON-LINEAR OSCILLATORS WITH LARGE PERTURBATIONS

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1. INTRODUCTION

The Melnikov method traditionally is restricted to study problems with weakly non-linear phenomena. This method developed by Melnikov [1] was generalized by Holmes and Marsden [2]. A detailed description of the Melnikov method was given by Guckenheimer and Holmes [3]. Similar results were obtained by Chow *et al.* [4]. For some oscillator systems with periodic perturbations it can be shown that, as a parameter is varied, repeated resonance of successive periods occur culminating in homoclinic or heteroclinic orbits. There exists a global method within the perturbation theory. The Melnikov function is used to measure the distance between unstable and stable manifolds when that distance is small by Guckenheimer and Holmes. It has been applied to problems that both the dissipation and forcing amplitude are small and the equation for the manifolds of the Hamiltonian for the undamped and unforced are known.

In this paper, the existence and bifurcation results are supplemented by a Melnikov method for strongly odd non-linear systems. The results imply that the homoclinic bifurcation is the limit of a countable sequence of subharmonic bifurcations.

We consider the response of strongly non-linear oscillators of the form

$$\ddot{u} + \varepsilon\eta\,\dot{u} + g_1(u) + \varepsilon g_2(u) = \varepsilon p f(\Omega t), \tag{1}$$

where dots are t derivatives. Here u is the displacement, $g_1(u)$ is a linear or an odd non-linear function of u; $g_2(u)$ is an odd non-linear function of u. The ordering parameter ε, η and p are assumed to be positive. Of interest are situations for which the response is strongly non-linear, which means that ε is not small, εp is not small, the response amplitude is large, or a combination of these.

In this work, we seek a new expansion parameter $\alpha = \alpha(\varepsilon)$ in such a way that a strongly non-linear system (large ε) is transformed into a weakly non-linear system (small α). The main ideas and analytical techniques to be used are briefly illustrated below:

(1) Consider the free vibration of the conservative non-linear oscillator governed by an equation of motion of the form

$$\ddot{u} + u + \varepsilon u^3 = 0, \tag{2}$$

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which does not involve perturbations. We use the results of a time transformation method [5–8] and obtain the following relation for the period T of the Duffing oscillator:

$$T = 2\pi\left(1 + \frac{3}{4}\varepsilon a^2\right)^{-1/2}\left[1 + \frac{3}{16}\alpha^2 + \frac{105}{1024}\alpha^4 + \sum_{n=3}^{\infty} D_{2n}\alpha^{2n}\right], \tag{3}$$

where a is the amplitude of the free vibration of equation (2),

$$D_{2n} = \frac{(2n - 1/2)(2n - 3/2)}{4n^2} D_{2n-2}, \quad D_4 = \frac{105}{1024}$$

and

$$\alpha = \frac{\varepsilon a^2}{(4 + 3\varepsilon a^2)}. \tag{4}$$

This relation is quickly convergent regardless of the magnitude of εa^2, since $\alpha < 1/3$ for all εa^2. This result indicates that for this oscillator an accurate period calculation can be obtained through consideration of the effect of only the fundamental harmonic of the non-linear oscillation. This has been pointed out by Jones [8].

(2) a parameter $\alpha = \alpha(\varepsilon, a)$ is defined [8] and

(3) the square frequency Ω^2 is expressed as a leading factor $(1 + 3\varepsilon a^2/4)$,

$$\Omega^2 = (1 + \tfrac{3}{4}\varepsilon a^2)(1 + \alpha\sigma). \tag{5}$$

We suppose that the forcing frequency Ω can be expressed using the parameter α defined for the free vibration equation (3). Since the problem is being approached in a backwards fashion (that is, normally Ω is specified and a is to be determined) the parameter α is assumed to be known ahead of time. Furthermore, we assume the leading factor $(1 + 3\varepsilon a^2/4)$ to represent a reasonably accurate approximation to the backbone curve [8], the curve defining the amplitude dependence of free vibration frequency.

For example, through time transform of the above these three steps, the Melnikov method can be used for systems with strongly odd non-linear terms and large perturbations.

The perturbed system is

$$\ddot{u} + \varepsilon\eta\,\dot{u} + Au + \varepsilon Bu^3 = \varepsilon p\cos\Omega t. \tag{6}$$

Of interest are situations for which the perturbations are strongly non-linear, which means that ε is not small, εp is not small, the response amplitude is large, or a combination of these. In this work, we seek a new expansion parameter $\alpha = \alpha(\varepsilon)$ in such a way that a strongly perturbed non-linear system (large ε) is transformed into a weakly perturbed non-linear system (small α). It results in

$$\ddot{u} + Cu + Du^3 + \alpha\left\{\left[\sigma + \frac{3}{A}(1 - A + \sigma A)\right]\ddot{u} + \frac{4\mu}{ABa^2}\dot{u} - \frac{4p}{ABa^2}\cos\Omega t\right\} = 0, \tag{7}$$

where $C = (1 - 3\alpha)$, $D = 4\alpha/Aa^2$, $\mu = \eta\Omega$ and α is a small parameter.

The Melnikov method can be used for large strongly odd perturbed non-linear terms and large perturbations systems, necessary through time transform.

2. APPLICATION TO A PENDULUM SUSPENDED ON A ROTATING ARM

The new structural design ([9] see Figure 1) in this paper permits the length ratio R/r, as a control parameter, of smaller than one. For various magnitudes of the length ratio and different relative positions (0 and π) of the pendulum to the arm, the abundance of regular and chaotic dynamics exhibit in the system, which is quite different from that of traditional vibrated pendulum. The shaft rotates about the y_1-axis with the angular rate ω. The pendulum is pivoted (axis A–A) on an arm rigidly attached to the shaft. This rotation of the pendulum is described by the angle θ. The gravitational vector is in the negative z_1 direction. The length of the arm is R and the length of the pendulum is r.

Motion is described by Lagrange's equation

$$mr^2\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + c_r\frac{\mathrm{d}u}{\mathrm{d}t} + \sin u(mRr\omega^2 + mr^2\omega^2\cos u + mrg\sin\omega t) = 0, \tag{8}$$

where ω is the angular rate of the shaft, c_r is the damping coefficient, g is the gravitational constant, m is the mass of the pendulum, t is time, r is the length of the pendulum, and R is the length of the arm which is the distance from the pivot to the center of the shaft. Let $\theta = u + \pi$. Thus, the equation of motion (8) becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - \omega^2\sin u(\rho - \cos u) + \varepsilon\left(2\eta\frac{\mathrm{d}u}{\mathrm{d}t} - f\sin u\sin\omega t\right) = 0, \tag{9}$$

where $2\eta = c_r/mr^2$, $\rho = R/r$ and $f = g/r$.

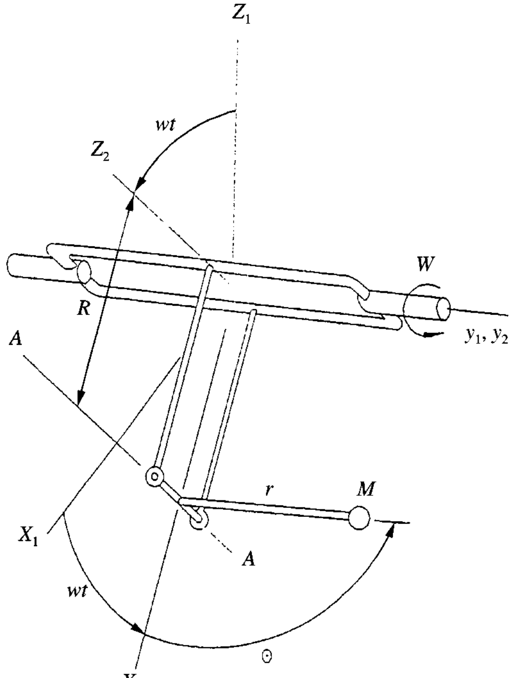

Figure 1. The pendulum on rotating arm.

By the time transform of the above three steps, a modified version of subharmonic Melnikov functions [10–12] is given by

$$M^{m/n}(T_0) = \int_0^{mT} \dot{u}_k(T)\left[\frac{3(1+\bar{A})+\sigma\bar{A}}{\bar{A}}\frac{\mathrm{d}^2u}{\mathrm{d}T^2} + \frac{4}{\bar{A}\bar{B}a^2}\left(\tilde{\mu}\frac{\mathrm{d}u}{\mathrm{d}T} - f\sin u \sin \varpi T\right)\right]\mathrm{d}T$$

$$= \frac{4}{\bar{A}\bar{B}a^2}\int_0^{mT}\dot{u}_k(T)[\tilde{\mu}\dot{u}_k - f\sin u_k \sin \varpi T]\,\mathrm{d}T$$

$$= \frac{4}{\bar{A}\bar{B}a^2}[\tilde{\mu}\times J_1(m,n) - f\times J_2(m,n)\sin \varpi T_0], \tag{10}$$

$$J_1(m,n) = \int_0^{mT}\dot{u}_k^2(T)\,\mathrm{d}T = \frac{2\tilde{A}^2k^2}{\tilde{B}(1+k^2)^2}\int_0^{mT} cn^2\left\{\sqrt{\left(\frac{\tilde{A}}{1+k^2}\right)}t\right\}\mathrm{d}n^2\left\{\sqrt{\left(\frac{\tilde{A}}{1+k^2}\right)}t\right\}\mathrm{d}T$$

$$= \frac{8\tilde{A}^2n}{3\tilde{B}(1+k^2)^{3/2}}[(k^2-1)K(k) + (1+k^2)E(k)]$$

where $K(k)$ is the complete elliptic integral of the first kind and

$$J_2(m,n) = \int_0^{mT}\dot{u}_k(T)\sin u_k(T)\sin \varpi T\,\mathrm{d}T$$

$$\cong \int_0^{mT}\dot{u}_k(T)\,u_k(T)\sin \varpi T\,\mathrm{d}T$$

$$= \begin{cases} 0 & n \neq 0 \text{ or } m = odd, \\ \dfrac{(\tilde{A})^{3/2}\pi^3m^2}{2\tilde{B}(1+k^2)K^2(k)}\cos ech\,\dfrac{\pi m K'(k)}{2\sqrt{\tilde{A}}K(k)} & n = 1 \text{ and } m = even. \end{cases}$$

If we define

$$R_m(m) = \frac{J_1(m,1)}{J_2(m,1)} = \frac{16\sqrt{\tilde{A}}K^2(k)[(k^2-1)K(k)+(1+k^2)E(k)]}{3(1+k^2)^{1/2}\pi^3m^2}\sinh\frac{\pi m K'(k)}{2\sqrt{\tilde{A}}K(k)}, \tag{11}$$

it is easy to see that the condition for the existence of zero's of the subharmonic Melnikov function becomes

$$\frac{f}{\eta} \geqslant 2\Omega R_m(\varpi). \tag{12}$$

The parameter values $f/\eta = 2\Omega R_m(\varpi)$ are critical in some sense, we call bifurcation. The bifurcation values of f/η are approximated by equation (12). Table 1 is related to the first appearance of certain subharmonics with short periods. As ρ increases such subharmonics arise from pitchfork bifurcations. In order to determine numerically these bifurcation value, we have to follow the corresponding subharmonic in the direction of decreasing ρ until it vanishes. For ρ values slightly above the critical value the change in ρ has to be carried out in steps of $\Delta\rho = 0{\cdot}001$ and $\Delta\varepsilon f = 0{\cdot}5$.

TABLE 1

Theoretical and numerical bifurcation values of ρ for subharmonics in dependence on ω, $\varepsilon\eta$ and εf in the strongly odd non-linear oscillators

m	ω	$\varepsilon\eta$	εf	Theoretical value	Numerical value
2	2·0	0·4	14·5	0·607	0·6062
2	2·0	0·4	15·0	0·480	0·4812
2	2·0	0·4	16·0	0·245	0·2453
2	2·0	0·4	16·5	0·133	0·1357

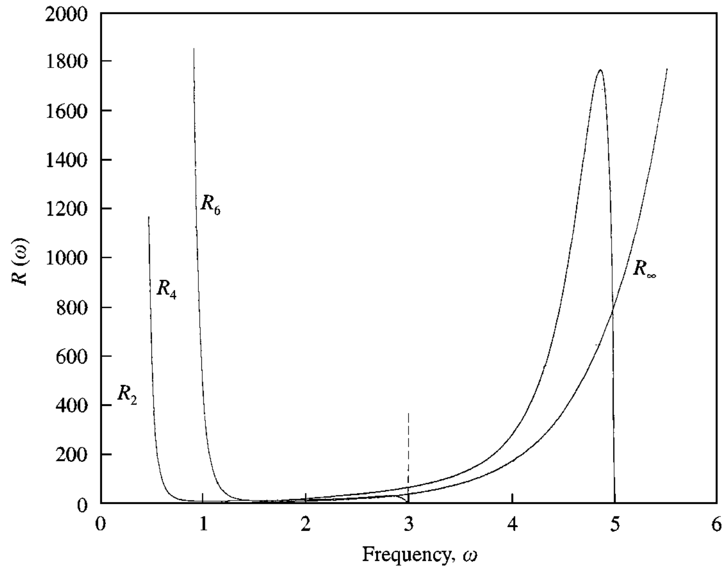

Figure 2. Bifurcation functions $R_m(\omega)$ in dependence upon ω from equation (13).

Using the relative $4\sqrt{1+k^2}K(k)/\sqrt{\tilde{A}}=2\pi m/\varpi$, the even subharmonic condition can be obtained,

$$R_m(\varpi) = \frac{4(\tilde{A})^{3/2}[(1+k^2)E(k) - (1-k^2)K(k)]}{3\pi\varpi^2(1+k^2)^{3/2}} \sinh\left(\frac{\varpi\sqrt{1+k^2}\,K'(k)}{\tilde{A}}\right). \tag{13}$$

For arbitrary fixed ϖ, we study the limit behavior of the subharmonic Melnikov functions for $m \to \infty$, i.e., $k \to 1$. Equation (13) is to verify that this limit exists

$$R_\infty(\varpi) = \frac{2\sqrt{2}\tilde{A}^{3/2}}{3\pi\varpi^2} \sinh\left(\frac{\varpi\sqrt{2}\pi}{2\tilde{A}}\right). \tag{14}$$

Also, using the expansion of $E(k)$ and $K(k)$ in $k \to 1$, for arbitrary given ϖ and enough k approach 1 (i.e., enough large m), we can prove

$$R_m(\varpi) < R_\infty(\varpi). \tag{15}$$

We conclude that for $f/\eta > 2\Omega R_\infty(\varpi)$, one have infinitely many periodic orbits. Moreover, in the limit $m \to \infty$, for resonance $T(k) \to \infty$, which implies that we are approaching the homoclinic orbit. This phenomenon we call chaos. When $\omega = \varpi$, then Figure 2 shows

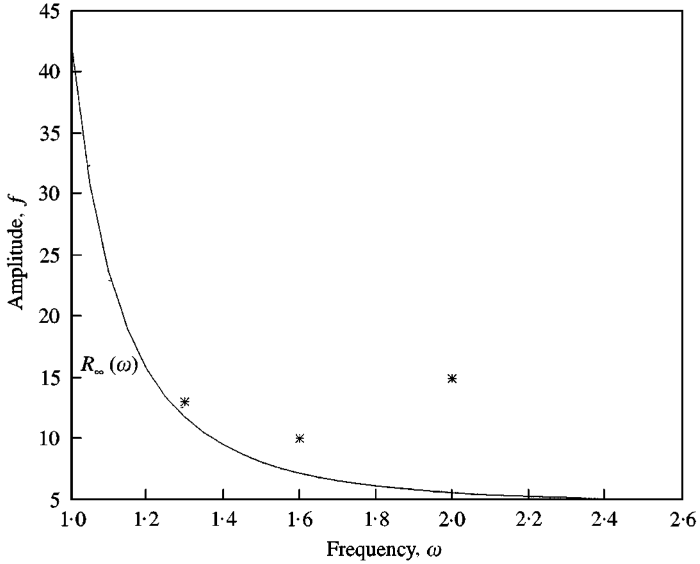

Figure 3. Regions of chaos in the forcing amplitude f versus the frequency ω plane from theory for equations (12) and (13) and numerical simulation for parameter values at $(\eta, \rho, \alpha, \varepsilon) = (0{\cdot}2, 0{\cdot}5, 1/4{\cdot}0, 2{\cdot}0)$.

bifurcation functions $R_m(\omega)$ in dependence upon ω. Only the functions with the largest m values can be close to $R_\infty(\omega)$. It is obvious that the $R_\infty(\omega)$, $R_6(\omega)$ curve ends at $\omega = 3$ and 5 respectively. The true three small bands (∗) of chaotic motion and the theoretical region satisfying equations (12) and (13) are drawn in the forcing amplitude f versus the frequency ω plane in Figure 3.

3. CONCLUSIONS

The Melnikov method is traditionally restricted to study problems with weakly non-linear phenomena including sufficient small harmonic excitation. This paper allows to extend Melnikov approach to specifically two-dimensional differential equations that possess strongly odd non-linear function of the displacement and are subjected to large harmonic excitation. It is evident that agreement between theory and numerical simulation can clarify this problem.

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REFERENCES

1. V. K. Melnikov 1963 *Transactions of the Moscow Mathematical Society* **12**, 1–57. On the stability of the center for time periodic perturbations.
2. P. J. Holmes and J. E. Marsden 1981 *Archive for Rational Mechanics and Analysis* **76**, 135–166. Partial differential equation with infinitely many periodic orbits: chaotic oscillations of forced beam.

3. M. GUCKENHEIMER and P. J. HOLMES 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York: Springer-Verlag.
4. S. N. CHOW, J. K. HALE and J. MALLET-PARET 1980 *Journal of Differential Equations* **37**, 351–373. In example Bifurcation to Homoclinic Orbits.
5. T. D. BURTON 1981 *International Journal of Non-Linear Mechanics* **17**, 7–19. Nonlinear oscillator limit cycle analysis using a time transformation approach.
6. T. D. BURTON and M. N. HAMDAN 1983 *Journal of Sound and Vibration* **87**, 543–554. Analysis of nonlinear autonomous conservative oscillators by a time transformation method.
7. T. D. BURTON 1984 *International Journal of Non-Linear Mechanics* **19**, 397–407. Perturbation method for certain nonlinear oscillators.
8. T. D. BURTON and Z. RAHMAN 1986 *International Journal of Non-Linear Mechanics* **21**, 135–146. In the multi-scale analysis of strongly nonlinear forced oscillators.
9. Z. M. GE and F. N. KU 1997 *Japanese Journal of Applied Physics* **36**, Part 1, 7052–7060. Stability, bifurcation and chaos of a pendulum on a rotating arm.
10. I. S. GRADSHTEYN and I. M. RYZHIK 1900 *Table of Integrals Series and Products* (Al. Jeffrey, editor) London: Academic Press, fifth edition.
11. P. F. BYRD and M. D. FRIEDMAN 1954 *Handbook of Elliptic Integrals for Engineers and Physicists*. Berlin, Gottingen, Heidelberg: Springer-Verlag.
12. Z. M. GE and F. N. KU 1997 *Japanese Journal of Applied Physics* **37**, Part 1, March. Melnikov method for strongly odd nonlinear oscillators.